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ECS452 2014/1 Part I.4 Dr.Prapun

3 An Introduction to Communication Systems Over Discrete Memoryless Channel (DMC)

Before: No consideration of the channel

Now: Try to represent the rest of the sys. as an equi. channel

In this section, we keep our analysis of the communication system simple by considering purely digital systems. To do this, we assume all non-source-coding parts of the system, including the physical channel, can be combined into an "equivalent channel" which we shall simply refer to in this section as the "channel".

3.1 Discrete Memoryless Channel (DMC) Models

Example 3.1. The **binary symmetric channel (BSC)**, which is the simplest model of a channel with errors, is shown in Figure 4.

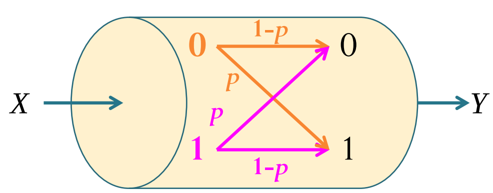


Figure 4: Binary symmetric channel and its channel diagram

- "Binary" means that the there are two possible values for the input and also two possible values for the output. We normally use the symbols 0 and 1 to represent these two values.
- Passing through this channel, the input symbols are complemented with probability p.

Ex: BSC

```
%% Simulation parameters
% The number of symbols to be transmitted
n = 20;
% Channel Input
S_X = [0 1]; S_Y = [0 1];
p_X = [0.3 0.7];
% Channel Characteristics
p = 0.1; Q = [1-p p; p 1-p];
```

>> BSC_demo

ans =

x = 1 0 1 1 1 1 1 1 1 1 1 1 0 1 0 1 1 1 1 1

ans =

y = 1 1 1 1 1 1 1 1 1 0 1 1 0 1 0 1 1 1 1 1

$P[X=0] \approx \frac{3}{20} = 0.15$

$P[X=1] \approx \frac{17}{20} = 0.85$

$P[Y=0] \approx \frac{3}{20} = 0.15$

$P[Y=1] \approx \frac{17}{20} = 0.85$

$P[Y=0 \mid X=0] \approx \frac{2}{3}$ $\quad \frac{2/20}{3/20}$

$P[Y=1 \mid X=0] \approx \frac{1}{3}$

$P[Y=0 \mid X=1] \approx \frac{1}{17} \approx 0.0588$

$P[Y=1 \mid X=1] \approx 16/17 \approx .9412$

p_X =

0.3000 0.7000

p_X_sim =

0.1500 0.8500

q =

0.3400 0.6600

q_sim =

0.1500 0.8500

Q =

0.9000 0.1000

0.1000 0.9000

Q_sim =

0.6667 0.3333

0.0588 0.9412

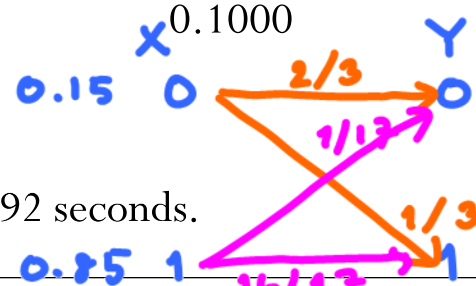
PE_sim =

0.1000

PE_theretical =

0.1000

$$Q \approx \begin{array}{c|cc} X \backslash Y & 0 & 1 \\ \hline 0 & 2/3 & 1/3 \\ 1 & 1/17 & 16/17 \end{array}$$



Elapsed time is 0.134992 seconds.

[BSC_demo.m]

Ex: BSC

```
%% Simulation parameters
% The number of symbols to be transmitted
n = 1e4;
% Channel Input
S_X = [0 1]; S_Y = [0 1];
p_X = [0.3 0.7];
% Channel Characteristics
p = 0.1; Q = [1-p p; p 1-p];
```

```
>> BSC_demo

p_X =
    0.3000    0.7000
p_X_sim =
    0.3037    0.6963
q =
    0.3400    0.6600
q_sim =
    0.3407    0.6593

Q =
    0.9000    0.1000
    0.1000    0.9000
Q_sim =
    0.9078    0.0922
    0.0934    0.9066
PE_sim =
    0.0930
PE_theretical =
    0.1000

Elapsed time is 0.922728 seconds.
```


[BSC_demo.m]

- It is simple, yet it captures most of the complexity of the general problem.

Definition 3.2. Our general model for discrete memoryless channel (DMC) is shown in Figure 5.

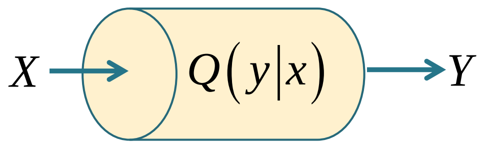


Figure 5: Discrete memoryless channel

BSC
$P[X=0] = p(0)$
$P[X=1] = p(1)$
$\underline{p} = [p(0) \quad p(1)]$

- The channel input is denoted by a random variable X.
  - The pmf $p_X(x)$ is usually denoted by simply $p(x)$ and usually expressed in the form of a row vector $\underline{p}$ or $\underline{\pi}$.
  - The support S_X is often denoted by $\mathcal{X}$.

BSC
$P[Y=0] = q(0)$
$P[Y=1] = q(1)$
$\underline{q} = [q(0) \quad q(1)]$

- Similarly, the channel output is denoted by a random variable Y.
  - The pmf $p_Y(y)$ is usually denoted by simply $q(y)$ and usually expressed in the form of a row vector $\underline{q}$.
  - The support S_Y is often denoted by $\mathcal{Y}$.
- The channel corrupts its input X in such a way that when the input is $X = x$, its output Y is randomly selected from the conditional pmf $p_{Y|X}(y|x)$. $= P[Y=y|X=x] = Q(y|x)$ channel transition probability
  - This conditional pmf $p_{Y|X}(y|x)$ is usually denoted by $Q(y|x)$ and usually expressed in the form of a probability transition matrix $\mathbf{Q}$:

matrix of transition probabilities

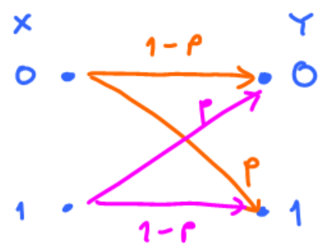
channel matrix

$$\mathbf{Q} = \begin{array}{c} \\ x \end{array} \begin{array}{c} y \\ \begin{bmatrix} \ddots & \vdots & \iddots \\ \cdots & P[Y=y|X=x] & \cdots \\ \iddots & \vdots & \ddots \end{bmatrix} \end{array}$$

$\frac{P(A \cap B)}{P(B)} = \frac{P[X=x, Y=y]}{P[X=x]}$

- The channel is called memoryless[9] because its channel output at a given time is a function of the channel input at that time and is not a function of previous channel inputs.
- Here, the transition probabilities are assumed constant. However, in many commonly encountered situations, the transition probabilities are time varying. An example is the wireless mobile channel in which the transmitter-receiver distance is changing with time.

Example 3.3. For a binary symmetric channel (BSC) defined in 3.1,



$$\mathbf{Q} = \begin{array}{c|cc} x \backslash y & 0 & 1 \\ \hline 0 & 1-p & p \\ 1 & p & 1-p \end{array}$$

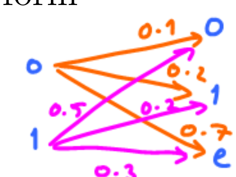
Example on the slide : $p = 0.1$

$$\mathbf{Q} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$$

Example 3.4. Suppose, for a DMC, we have $\mathcal{X} = \{x_1, x_2\}$ and $\mathcal{Y} = \{y_1, y_2, y_3\}$. Then, its probability transition matrix $\mathbf{Q}$ is of the form

$$\mathbf{Q} = \begin{bmatrix} Q(y_1|x_1) & Q(y_2|x_1) & Q(y_3|x_1) \\ Q(y_1|x_2) & Q(y_2|x_2) & Q(y_3|x_2) \end{bmatrix}.$$

$$\begin{array}{c|ccc} x \backslash y & 0 & 1 & e \\ \hline 0 & 0.1 & 0.2 & 0.7 \\ 1 & 0.5 & 0.2 & 0.3 \end{array}$$



You may wonder how this $\mathbf{Q}$ happens in real life. Let's suppose that the input to the channel is binary; hence, $\mathcal{X} = \{0, 1\}$ as in the BSC. However, in this case, after passing through the channel, some bits can be lost[10] (rather than corrupted). In such case, we have three possible outputs of the channel: 0, 1, e where the "e" represents the case in which the bit is erased by the channel.

[9]Mathematically, the condition that the channel is memoryless may be expressed as [10, Eq. 6.5-1 p. 355]

$$p_{X_1^n|Y_1^n}(x_1^n|y_1^n) = \prod_{k=1}^{n} Q(y_k|x_k).$$

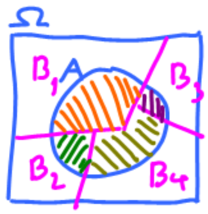
[10]The receiver knows which bits have been erased.

3.5. Knowing the input probabilities $\underline{p}$ and the channel probability transition matrix $\mathbf{Q}$, we can calculate the output probabilities $\underline{q}$ from

$$\underline{q} = \underline{p}\mathbf{Q}.$$

To see this, recall the **total probability theorem**: If a (finite or infinitely) countable collection of events $\{B_1, B_2, \ldots\}$ is a partition of Ω, then

$$P(A) = \sum_i P(A \cap B_i) = \sum_i P(A|B_i)P(B_i). \tag{5}$$



$$P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) + P(A \cap B_4)$$

$$= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3) + P(A|B_4)P(B_4)$$

$$P(A|B) \equiv \frac{P(A \cap B)}{P(B)}$$

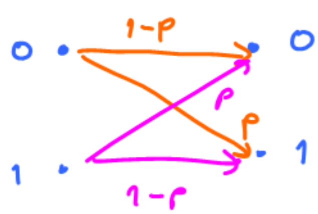
by definition

For us, event A is the event $[Y = y]$. Applying this theorem to our variables, we get

$$q(y) = P[Y = y] = \sum_x P[X = x, Y = y]$$
$$= \sum_x P[Y = y|X = x]\, P[X = x] = \sum_x Q(y|x)p(x).$$

This is exactly the same as the matrix multiplication calculation performed to find each element of $\underline{q}$.

Example 3.6. For a binary symmetric channel (BSC) defined in 3.1,



$$\overset{A}{P[Y=0]} = \overset{A \cap B_1}{\underset{P(A|B_1)P(B_1)}{P[Y=0, X=0]}} + \overset{A \cap B_2}{\underset{P(A|B_2)P(B_2)}{P[Y=0, X=1]}}$$

total probability theorem

$$= P[Y=0|X=0]\,P[X=0] + P[Y=0|X=1]\,P[X=1]$$

$$= (1-p)p(0) + p\,p(1) = [p(0) \quad p(1)] \begin{bmatrix} 1-p \\ p \end{bmatrix}$$

Try this @ home

$$P[Y=1] = p \times p(0) + (1-p)\,p(1) = [p(0) \quad p(1)] \begin{bmatrix} p \\ 1-p \end{bmatrix}$$

$$\underline{q} = [q(0) \quad q(1)] = [p(0) \quad p(1)] \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$$

$$[a][b \quad c]$$
$$[ab \quad ac]$$

$$= \underline{p}\,\mathbf{Q}$$

3.7. Recall, from ECS315, that there is another matrix called the **joint probability matrix P**. This is the matrix whose elements give the joint probabilities $P_{X,Y}(x,y) = P[X = x, Y = y]$:

$$\mathbf{P} = \begin{array}{c c} & y \\ x & \begin{bmatrix} \ddots & \vdots & \cdot^{\cdot^{\cdot}} \\ \cdots & P[X = x, Y = y] & \cdots \\ \cdot^{\cdot^{\cdot}} & \vdots & \ddots \end{bmatrix} \end{array}.$$

Recall also that we can get $p(x)$ by adding the elements of **P** in the row corresponding to x. Similarly, we can get $q(y)$ by adding the elements of **P** in the column corresponding to y.

By definition, the relationship between the conditional probability $Q(y|x)$ and the joint probability $P_{X,Y}(x,y)$ is

$$Q(y|x) = \frac{P_{X,Y}(x,y)}{p(x)}.$$

Equivalently,

$$P_{X,Y}(x,y) = p(x)Q(y|x).$$

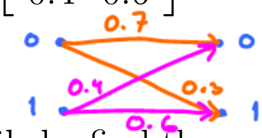
Therefore, to get the matrix **P** from matrix **Q**, we need to multiply each row of **Q** by the corresponding $p(x)$. This could be done easily in MATLAB by first constructing a diagonal matrix from the elements in $\underline{p}$ and then multiply this to the matrix **Q**:

$$\mathbf{P} = \left(\operatorname{diag}\left(\underline{p}\right)\right)\mathbf{Q}.$$

Example 3.8. Binary A~~s~~symmetric Channel (BAC): Consider a binary input-output channel whose matrix of transition probabilities is

$$\mathbf{Q} = \begin{array}{c c} x \backslash y & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \end{array}.$$

(a) Draw the channel diagram.



$P[X=0] = P[X=1] = 0.5$ $\qquad$ $q(0), q(1)$

(b) If the two inputs are equally likely, find the corresponding output probabilities and the joint probability matrix **P** for this channel.

[16, Ex. 11.3]

$$\underline{q} = \underline{p}\mathbf{Q} = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}\begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.55 & 0.45 \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \begin{array}{l} \xrightarrow{\times p(0)} \\ \xrightarrow{\times p(1)} \end{array} = \mathbf{P} = \begin{array}{c c} x \backslash y & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0.35 & 0.15 \\ 0.2 & 0.3 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} \downarrow + & \downarrow + \\ 0.55 & 0.45 \\ q(0) & q(1) \end{array}$$